Lax pair formulation for the one-dimensional Heisenberg ***XXZ*** open chain

1996 J. Phys. A: Math. Gen. 29 L489

(http://iopscience.iop.org/0305-4470/29/18/007)

J. Phys. A: Math. Gen. **29** (1996) L489–L492. Printed in the UK

LETTER TO THE EDITOR

Lax pair formulation for the one-dimensional Heisenberg XXZ open chain

Huan-Qiang Zhou†

CCAST(World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
and
Department of Physics, Chongqing University, Chongqing, Sichuan 630044, People's Republic of China‡

Received 11 January 1996

Abstract. The Lax pair formulation is presented for completely integrable quantum lattice spin open chains. Specifically, the Lax pair for the one-dimensional Heisenberg XXZ open chain is explicitly constructed. Our construction provides an alternative and direct demonstration of the quantum integrability of the system.

Recently there has been considerable interest in the study of completely integrable lattice spin open chains [1–8]. As was shown by Sklyanin [1], there is a variant of the usual formalism of the quantum inverse scattering method (QISM) [9–11], which may be used to describe systems on a finite interval with independent boundary conditions at each end. Central to his approach is the introduction of a new algebraic structure called the reflection equations (RE) [12]. Although Sklyanin's argument was carried out only for the P- and T-invariant R matrices, it is now known that his formalism may be extended to apply to any open chains integrable by the quantum R matrix approach [8]. Much attention has since been paid to the solutions of the RE which present boundary K matrices compatible with the integrability condition. Recently, boundary K matrices have been constructed by de Vega and González Ruiz [6] for the Heisenberg spin chain and by the present author [7, 8] for the one-dimensional (1D) Hubbard chain and the 1D Bariev chain.

On the other hand, the traditional basis for applying QISM to a completely integrable system is to represent the equations of motion of the system in Lax form. Following Izergin and Korepin [9, 10], one may show that for systems with periodic boundary conditions, the existence of the quantum R matrix allows one to express the original equations of motion in Lax form. In particular, the Lax pairs for a variety of physically interesting models were given in [13–16]. Thus, one may expect that there is a variant of the usual Lax pair formulation for describing quantum integrable lattice spin open chains.

The aim of this letter is to present the Lax pair for the 1D Heisenberg XXZ open chain in explicit form. Our construction provides an alternative description for the quantum integrability of the system. Indeed, the boundary K matrices thus constructed are consistent with those obtained through solving the RE with the given R matrix.

† E-mail address:hqzhou@cqu.edu.cn
‡ Mailing address.

0305-4470/96/180489+04$19.50

The Lax pair formulation may be described as follows. Instead of directly considering the equations of motion, let us study an operator version of an auxiliary linear problem:

$$\begin{aligned} &\Psi_{j+1} = L_j(\lambda)\Psi_j \qquad j = 1, 2, \ldots, N \\ &\frac{\mathrm{d}}{\mathrm{d}t}\Psi_j = M_j(\lambda)\Psi_j \qquad j = 2, \ldots, N \\ &\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{N+1} = \delta_N(\lambda)\Psi_{N+1} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 = \delta_1'(\lambda)\Psi_1 . \end{aligned} \tag{1}$$

Here L_j, M_j, δ and δ' are some matrices depending on the spectral parameter λ and the dynamical variables. The consistency conditions for (1) yield the Lax equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}L_j(\lambda) = M_{j+1}(\lambda)L_j(\lambda) - L_j(\lambda)M_j(\lambda) \qquad j = 2, \ldots, N-1 \\ &\frac{\mathrm{d}}{\mathrm{d}t}L_N(\lambda) = \delta_N(\lambda)L_N(\lambda) - L_N(\lambda)M_N(\lambda) \\ &\frac{\mathrm{d}}{\mathrm{d}t}L_1(\lambda) = M_2(\lambda)L_1(\lambda) - L_1(\lambda)\delta_1'(\lambda). \end{aligned} \tag{2}$$

A lattice spin open chain is called completely integrable if we can express the equations of motion in the Lax form (2). In fact, it is readily shown that a transfer matrix

$$\tau(\lambda) = \mathrm{Tr}(K_+(\lambda)L_N(\lambda)\cdots L_1(\lambda)K_-(\lambda)L_1^{-1}(-\lambda)\cdots L_N^{-1}(-\lambda)) \tag{3}$$

does not depend on time t, provided the following constraints hold:

$$K_-(\lambda)\delta_1'(-\lambda) = \delta_1'(\lambda)K_-(\lambda) \tag{4}$$

and

$$\mathrm{Tr}[K_+(\lambda)\delta_N(\lambda)A_N(\lambda)] = \mathrm{Tr}[K_+(\lambda)A_N(\lambda)\delta_N(-\lambda)] \tag{5}$$

with

$$A_N(\lambda) = L_N(\lambda)\cdots L_1(\lambda)K_-(\lambda)L_1^{-1}(\lambda)\cdots L_N^{-1}(\lambda). \tag{6}$$

This implies that the system possesses an infinite number of conserved quantities.

As an application, let us consider the 1D Heisenberg XXZ open chain with the Hamiltonian [1]

$$\begin{aligned} H = -\sum_{j=2}^{N}\left[(\sigma_j^+\sigma_{j-1}^- + \sigma_j^-\sigma_{j-1}^+) + \frac{1}{2}\cos(2\eta)\sigma_j^z\sigma_{j-1}^z\right] + \frac{1}{2}\sin(2\eta)\cot\xi_+\sigma_N^z \\ +\frac{1}{2}\sin(2\eta)\cot\xi_-\sigma_1^z. \end{aligned} \tag{7}$$

Here $\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$ and $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are the usual Pauli spin operators at a lattice site j, η is a parameter associated with the anisotropy of the Hamiltonian (7) and the $\xi_\pm$ are some constants describing the boundary effects.

It is not difficult to check that the equations of motion derived from the Hamiltonian (7) may be cast in the Lax form (2). Indeed, in our case, the L and M matrices take the form

$$L_j(\lambda) = \begin{pmatrix} \sin(\lambda+\eta)\cos\eta + \sin\eta\cos(\lambda+\eta)\sigma_j^z & \sin 2\eta\sigma_j^- \\ \sin 2\eta\sigma_j^+ & \sin(\lambda+\eta)\cos\eta - \cos(\lambda+\eta)\sin\eta\sigma_j^z \end{pmatrix} \tag{8}$$

and

$$M_j(\lambda) = \begin{pmatrix} f\sigma_j^+\sigma_{j-1}^- + g\sigma_j^-\sigma_{j-1}^+ - d\sigma_j^z\sigma_{j-1}^z & p(\sigma_j^-\sigma_{j-1}^z - \sigma_j^z\sigma_{j-1}^-) \\ +d(\sigma_j^z + \sigma_{j-1}^z) & +q(\sigma_j^- + \sigma_{j-1}^-) \\ -p(\sigma_j^+\sigma_{j-1}^z - \sigma_j^z\sigma_{j-1}^+) & g\sigma_j^+\sigma_{j-1}^- + f\sigma_j^-\sigma_{j-1}^+ - d\sigma_j^z\sigma_{j-1}^z \\ +q(\sigma_j^+ + \sigma_{j-1}^+) & -d(\sigma_j^z + \sigma_{j-1}^z) \end{pmatrix} \tag{9}$$

where

$$f = -\mathrm{i}\frac{2\cos(\lambda+\eta)\sin\eta}{\sin(\lambda+2\eta)} \qquad g = \mathrm{i}\frac{2\cos(\lambda - \sin\eta}{\sin(\lambda-2\eta)}$$

$$d = -\frac{\mathrm{i}}{4}\frac{\sin 4\eta\sin 2\eta}{\sin(\lambda+2\eta)\sin(\lambda-2\eta)} \qquad p = -\frac{\mathrm{i}}{2}\frac{\sin 4\eta\sin\lambda}{\sin(\lambda+2\eta)\sin(\lambda-2\eta)}$$

$$q = -\mathrm{i}\frac{\cos\lambda\sin 2\eta\sin 2\eta}{\sin(\lambda+2\eta)\sin(\lambda-2\eta)} \,.$$

From equation (2), it follows that

$$\delta_1'(\lambda) = \frac{\mathrm{i}\sin^2 2\eta}{\sin(\lambda+2\eta)\sin(\lambda-2\eta)\sin\xi_-} \times\begin{pmatrix} -\frac{1}{2}\sin(2\eta+\xi_-)\sigma_1^z + \frac{1}{2}\sin(2\eta)\cos\xi_- & \sin(\lambda-\xi_-)\sigma_1^- \\ -\sin(\lambda+\xi_-)\sigma_1^+ & -\frac{1}{2}\sin(2\eta-\xi_-)\sigma_1^z - \frac{1}{2}\sin(2\eta)\cos\xi_- \end{pmatrix} \tag{10}$$

and

$$\delta_N(\lambda) = \frac{i\sin^2 2\eta}{\sin(\lambda+2\eta)\sin(\lambda-2\eta)\sin\xi_+} \times\begin{pmatrix} -\frac{1}{2}\sin(2\eta+\xi_+)\sigma_1^z + \frac{1}{2}\sin(2\eta)\cos\xi_+ & -\sin(\lambda+\xi_+)\sigma_N^- \\ \sin(\lambda-\xi_+)\sigma_N^+ & -\frac{1}{2}\sin(2\eta-\xi_+)\sigma_N^z - \frac{1}{2}\sin(2\eta)\cos\xi_+ \end{pmatrix} . \tag{11}$$

We now proceed to study the constraint conditions (4) and (5). Setting

$$K_-(\lambda) = \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \beta(\lambda) \end{pmatrix} \tag{12}$$

and substituting this in equation (4), one may get

$$\frac{\alpha(\lambda)}{\beta(\lambda)} = -\frac{\sin(\lambda-\xi_-)}{\sin(\lambda+\xi_-)} \,. \tag{13}$$

Thus we have determined the boundary matrix $K_-(\lambda)$:

$$K_-(\lambda) = -\frac{1}{\sin\xi_-}\begin{pmatrix} \sin(\lambda-\xi_-) & 0 \\ 0 & -\sin(\lambda+\xi_-) \end{pmatrix} \tag{14}$$

(up to an unimportant scalar factor). In order to determine the boundary matrix $K_+(\lambda)$, let us first note that

$$A_N(\lambda) = L_N(\lambda)A_{N-1}(\lambda)L_N^{-1}(-\lambda) \,. \tag{15}$$

Obviously, the matrix elements of A_{N-1} commute with those of L_N. Keeping this fact in mind and noting that the matrix elements of A_{N-1} are independent, we immediately obtain

$$K_+(\lambda) = \begin{pmatrix} -\sin(\lambda + 2\eta - \xi_+) & 0 \\ 0 & \sin(\lambda + 2\eta + \xi_+) \end{pmatrix} \tag{16}$$

(up to an unimportant scalar factor). Evidently, our conclusion is consistent with that given by Sklyanin [1].

In conclusion, we have presented the Lax pair formulation for completely integrable quantum lattice spin open chains. As an application, we have constructed the Lax pair for the 1D Heisenberg XXZ open chain. The boundary K matrices thus constructed are consistent with those obtained using Sklyanin's formalism [1]. Thus our construction provides an alternative description for the quantum integrability of the system. The extension of our results to other open chains is straightforward. In particular, the formulation is applicable to any integrable periodic chains whose equations of motion may be cast into the Lax form. Combining this with Korepin and Izergin's work [9, 10], one is led to conclude that Sklyanin's formalism may be extended to apply to any systems integrable by the quantum R-matrix approach [7, 8].

This work was supported in part by the National Natural Science Foundation of China under grant No 19505009. I am grateful to Professors Fu-Cho Pu and Min-Guang Zhao for their support and encouragement. Thanks are also due to Mr Jing-Song He for his helpful discussions.

References

[1] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[2] Mezincescu L, Nepomechie R I and Rittenberg V 1990 *Phys. Lett.* **147A** 70
[3] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[4] Destri C and de Vega H J 1992 *Nucl. Phys.* B **374** 692
[5] Mezincescu L and Nepomechie R I 1991 *J. Phys. A: Math. Gen.* **24** L17
[6] de Vega H J and González Ruiz A 1991 *J. Phys. A: Math. Gen.* **26** L519
[7] Zhou H Q 1996 *Phys. Rev.* B **54** 41
[8] Zhou H Q 1996 *Phys. Rev.* B in press
[9] Korepin V E, Izergin G and Bogoliubov N M 1992 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[10] Faddeev L D 1984 *Les Houches 1982* ed J B Zuber and R Stora (Amsterdam: North-Holland); 1980 *Sov. Sci. Rev. Math. Phys.* C **1** 107
Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* vol 151 (Berlin: Springer) p 61
Izergin A G and Korepin V E 1982 *Sov. J. Part. Nucl. Phys.* **13** 20
[11] Thacker H B 1982 *Rev. Mod. Phys.* **53** 253
[12] Cherednik I 1984 *Theor. Math. Phys.* **61** 35
[13] Sogo K and Wadati M 1982 *Prog. Theor. Phys.* **68** 85
[14] Barouch E 1984 *Stud. Appl. Math.* **70** 151
[15] Wadati M, Olmedilla E and Akutsu Y 1987 *J. Phys. Soc. Japan* **56** 1340
[16] Zhou H Q, Jiang L J and Tang J G 1990 *J. Phys. A: Math. Gen.* **23** 213